Infinite Infrared Regularization in Krein Spaces

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5th March 2003

Talk presented at the
AMS Southeastern Section Meeting 14th–16th March 2003
Special Section
“Asymptotic Analysis, Stability, and Generalized Functions”
Baton Rouge, Louisiana
on the 15th March 2003
by

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The massless scalar field in $1+1$ dimensions has the two point function

$$\mathfrak{W}_2(x,y) = -D^+(x-y), \quad -D^+(x) \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0} \frac{1}{4\pi} \ln\left(x^2 + \mathrm{i}\varepsilon x^0\right),$$

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The tempered distribution

$$\left(\frac{1}{p}\right)_+ (\widetilde{f}) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{d}p\, \frac{\widetilde{f}(p) - \widetilde{f}(0)}{p}$$

is **not positive definite** in general, when $\widetilde{f}(0) \neq 0$.

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Locality: Whenever $(x_j - x_{j+1})^2 < 0$, then

$$\mathfrak{W}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = \mathfrak{W}(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n).$$

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Positivity: For $\underline{f}, \underline{g} \in \underline{\mathscr{S}}$, the **sesquilinear form** $\left(\underline{f}\,, \underline{g}\right) \equiv \sum_{m,n} \mathfrak{W}_{m+n}(f_n^* \otimes g_m)$
shall be positive, *i.e.*, $\left(\underline{f}\,, \underline{f}\right) \geq 0$.

Here, $f_n \otimes g_m(x_1, \ldots, x_{m+n}) \stackrel{\text{def}}{=} f_n(x_1, \ldots, x_n) g_m(x_{n+1}, \ldots, x_{m+n})$, and
$\underline{\mathscr{S}} \stackrel{\text{def}}{=} \left\{\underline{f} = (f_0, f_1, \ldots) | f_0 \in \mathbb{C},\ f_i \in \mathscr{S}(\mathbb{R}^{4i}),\ \#\{i, f_i \neq 0\} < \infty\right\}$.

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The mechanism behind this phenomenon is the presence **virtual states**, *i.e.*, physical equivalence classes of the potential, in theories with nontrivial gauge invariance [Bogolubov *et al.* 1990, Chapter 10].

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To carry out 3. and 4. one replaces positivity by the weaker

Hilbert space structure condition (HS). Assume that there exist Hilbert seminorms $p_n = (.\,,.)^{1/2}$ on $\mathscr{S}(\mathbb{R}^{4n})$ such that $\left|\mathfrak{W}_{n+m}(f_n^* \otimes g_m)\right| \leq p_n(f_n)p_m(g_m)$ for all $f_n \in \mathscr{S}(\mathbb{R}^{4n})$, $g_m \in \mathscr{S}(\mathbb{R}^{4m})$.

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Let $\mathcal{V}$ be a vector space equipped with an indefinite inner product $\langle . \, , . \rangle$.
A locally convex topology τ on $\mathcal{V}$ is called **majorant**,
if the inner product $\langle . \, , .. \rangle$ is jointly τ–continuous.
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Lemma (A). *i) To every majorant exists a weaker one defined by a single seminorm.*
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Proposition (B). *If a majorant is normed, i.e., defined by* $\|.\| \stackrel{\text{def}}{=} (.\,,.)^{1/2}$
then exists the ***metric (Gram) operator****:*

$$\langle x\,,y\rangle = (x\,,Jy), \quad \forall x,y \in \mathcal{K}.$$

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A–D render HS effective: They allow the construction of the largest Krein space closure to a given indefinite inner product, and the separation of the negative part, to arrive at a Hilbert space containing all physical information.

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$$\mathscr{S}_\alpha(\mathbb{R}^n) \stackrel{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^n) \;\middle|\; \forall k, q \in \mathbb{N}_0 : \sup_{x, |\sigma| \leq q} |x^k D_x^\sigma f(x)| \leq C_q A^k k^{k\alpha} \right\},$$

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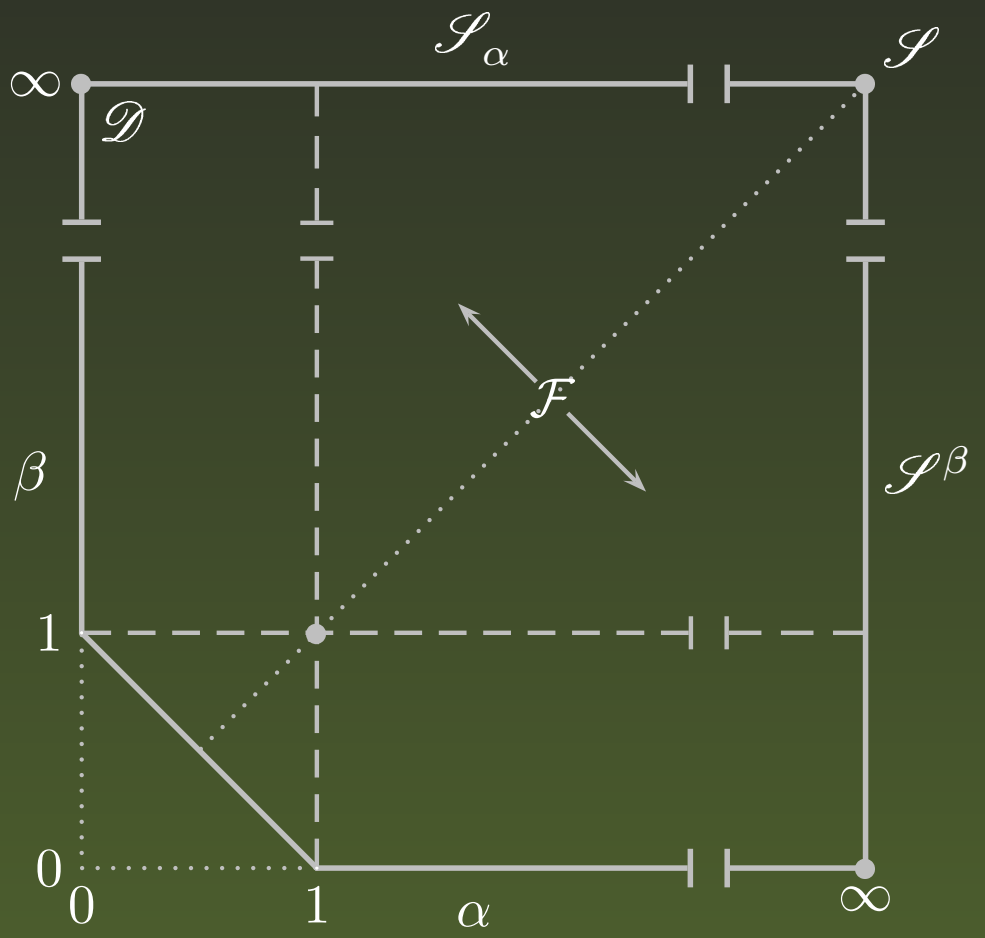
- Old acquaintance: **Schwartz space** $\mathscr{S} \cong \varinjlim_{\alpha,\beta\to\infty} \mathscr{S}_\alpha^\beta$;

  Compactly supported test functions $\mathscr{D} \cong \mathscr{S}_0(\mathbb{R}^n)$;

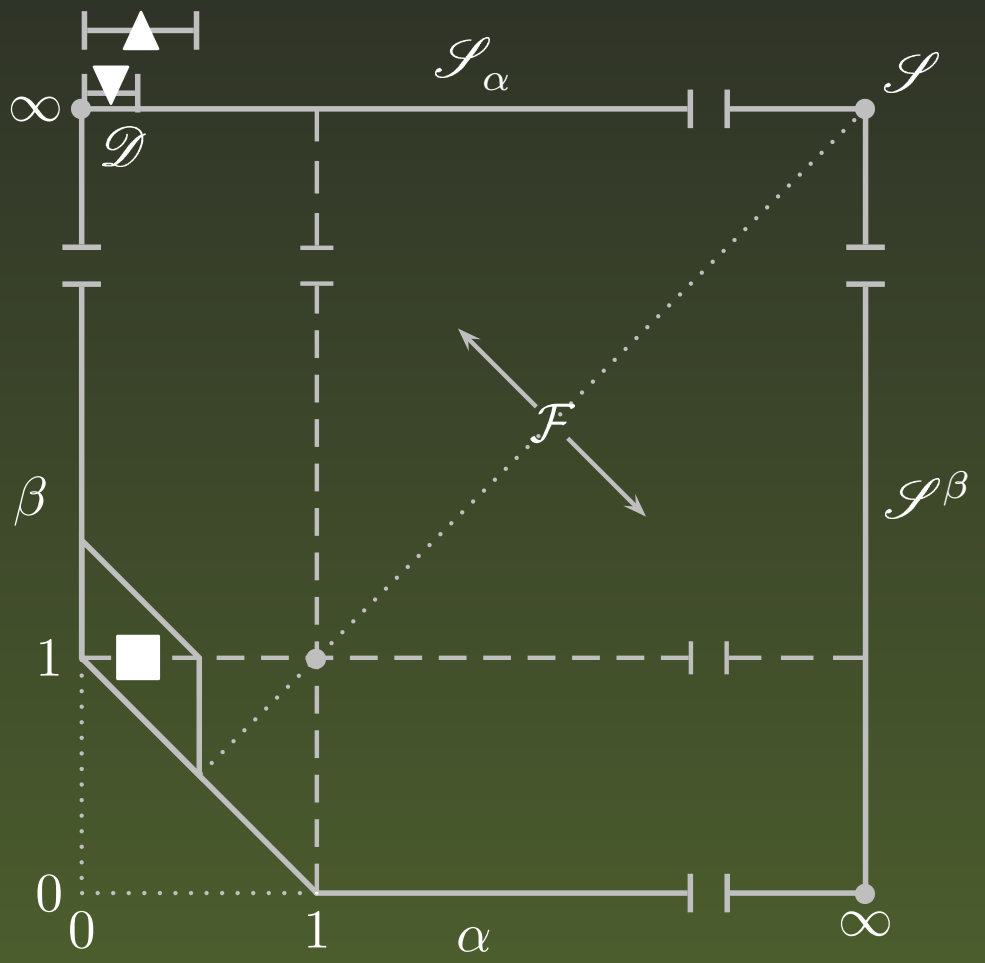
  Test functions for **Fourier hyperfunctions** $\mathscr{P}_* \cong \mathscr{S}_1^1$.

Strength of Singularity: Examples



Strength of Singularity: Examples

Suitable test function spaces for certain QFT models [Moschella/Strocchi 1992].



▲ The two-point function of the $1+1$-*dimensional dipole field*, solution of $\Box^2\varphi = 0$, is in ${\mathscr{S}_\alpha}'$, for $\alpha < 1/2$.

▼ The *vertex operator* $:\exp(\mathrm{i}g\varphi):$ of this field is in ${\mathscr{S}_\alpha}'$, for $\alpha < 1/4$.

■ In the temporal variables, $:\exp(\mathrm{i}g\varphi):$ for the dipole field in $3+1$ dimensions is in $\mathscr{S}_\alpha^{\beta\prime}$, $\alpha+\beta < 3/2$, $\alpha < 1/2$.

A Model with Infinite Infrared Singularity

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$$\langle f\,,g\rangle \stackrel{\text{def}}{=} (f\,,g)_{L^2} - \sum_{k=0}^{\infty} c_k^2 \overline{f}^{(k)}(0) g^{(k)}(0), \quad c_k \in \mathbb{R}\setminus\{0\},$$

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is an **indefinite product**, characterized by the **infra-exponential symbol of order** $1/(2\delta)$

$$J(\xi) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} c_k^2 \xi^k, \quad |J(\xi)| \leq C_\varepsilon \mathrm{e}^{\varepsilon|\xi|^{1/(2\delta)}}, \ \forall \varepsilon > 0.$$

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is an **indefinite product**, characterized by the **infra-exponential symbol of order** $1/(2\delta)$

$$J(\xi) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} c_k^2 \xi^k, \quad |J(\xi)| \leq C_\varepsilon \mathrm{e}^{\varepsilon|\xi|^{1/(2\delta)}}, \ \forall \varepsilon > 0.$$

It is well defined on $\mathscr{S}_\alpha^\beta(\mathbb{R})$ for any α and $\beta < \delta$ ($\beta \leq \delta$ suffices), since then

$$|c_k^2| \leq C \frac{\theta^k}{D^k \mathrm{e}^{2k\delta} k^{2k\delta}}, \quad \forall D > 0\ \exists \theta \in (0,1), C > 0,$$

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Neutral Decomposition

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Set $\gamma_k \stackrel{\text{def}}{=} k^{k\delta}$ and construct neutral functions $\mathcal{N} \stackrel{\text{def}}{=} \text{span}\{\chi_k\}_{k\in\mathbb{N}_0} \subset \mathscr{S}^{\beta}_{\alpha}$ satisfying

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defines a majorant τ on $\left\{\mathscr{S}_\alpha^{\beta<\delta}, \langle .\,,.\rangle\right\}$ through the seminorm

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$$(f\,,g) \stackrel{\text{def}}{=} \langle f^+\,,g^+\rangle + \sum_{i=0}^{\infty}\left\{\langle f\,,\chi_i\rangle\langle\chi_i\,,g\rangle + \overline{f^i}g^i\right\}, \quad \forall f,g\in\mathcal{K},$$

where $f^+ \stackrel{\text{def}}{=} \lim_{N\to\infty} f^{N+}$ (convergent in $\mathcal{K}$).

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In particular, setting $\chi_i\stackrel{\text{def}}{=}\gamma_i\widetilde{\chi}_i$, and $v^i=\widetilde{v}^i/\gamma_i$, one has

$$p(v)^2\ \stackrel{\text{def}}{=}\lim_{N\to\infty}\left[\langle v^{N+}\,,f^{N+}\rangle+\sum_{i=0}^{N}\left\{|\langle v\,,\chi_i\rangle|^2+|v^i|^2\right\}\right]$$

Sufficient Conditions for Regularization

Theorem. *Let $\mathcal{V}$ be a space with non-degenerate, indefinite inner product $\langle .\,,.\rangle$.*

0) There exists an orthogonal system $\{\widetilde{\chi}_i\}_{i\in\mathbb{N}_0}$ of mutually linearly independent, neutral vectors in $\mathcal{V}$.

1) For all $v\in\mathcal{V}$, the unique decomposition $v=\widetilde{v}^{N+}+\sum_{i=0}^{N}\widetilde{v}^i\widetilde{\chi}_i,\quad \widetilde{v}^i\in\mathbb{C},\ N\in\mathbb{N}_0$,

becomes asymptotically positive in the sense that $0\leq\lim_{N\to\infty}\langle\widetilde{v}^{N+}\,,\widetilde{v}^{N+}\rangle$.

2) There exists a sequence of complex numbers $\{\gamma_i\}_{i\in\mathbb{N}_0}$ such that both sequences $\{\gamma_i\langle\widetilde{\chi}_i\,,v\rangle\}$ and $\{\widetilde{v}^i/\gamma_i\}$ are in $l^2(\mathbb{N}_0)$.

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References

Books:

[Streater/Wightman 1964] *PCT, Spin and Statistics, and All That*, W. A. Benjamin.

[Bogolubov *et al.* 1990] *General Principles of Quantum Field Theory*, Kluwer.

[Strocchi 1993] *Selected Topics on the General Properties of Quantum Field Theory*, World Scientific.

[Bognár 1974] *Indefinite Inner Product Spaces*, Springer-Verlag.

[Azizov/Iokhvidov 1989] *Linear Operators in Spaces with an Indefinite Metric*, John Wiley & Sons.

Singularities and Functional Regularization in QFT:

[Wightman 1981] *The Choice of Testfunctions in Quantum Field Theory*, Journal of Mathematical Analysis and Applications **7B**, 769–791.

[Moschella/Strocchi 1992] *The Choice of Test Functions in Gauge Quantum Field Theory*, Letters in Mathematical Physics **24**, 103–113.

[Brüning/Nagamachi 1989] *Hyperfunction Quantum Field Theory: Basic Structural Results*, Journal of Mathematical Physics **30**, 2340–2359.

Indefiniteness and its Healing:

[Dubin/Tarski 1966] *Indefinite Metric Resulting from Regularization in the Infrared Region*, Journal of Mathematical Physics **7**, 574–577.

[Morchio *et al.* 1990] *Infrared and Vacuum Structure in Two-Dimensional Local Quantum Field Theory*, Journal of Mathematical Physics **31**, 1467–1477.

[AUS 1997] *Mathematical Problems of Gauge Quantum Field Theory: A Survey of the Schwinger Model*, Universitatis Iagellonicae Acta Mathematica **Fasciculus XXXIV**, 113–134, hep-th/9707166.

[AUS 2002] *Infinite Infrared Regularization and a State Space for the Heisenberg Algebra*, Journal of Mathematical Physics **43**, 243–259, math-ph/0105042.