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Zeno dynamics of von Neumann algebras

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Abstract
The dynamical quantum Zeno effect is studied in the context of von Neumann algebras. We identify a localized subalgebra on which the Zeno dynamics acts by automorphisms. The Zeno dynamics coincides with the modular dynamics of that subalgebra, if an additional assumption is satisfied. This relates the modular operator of that subalgebra to the modular operator of the original algebra by a variant of the Kato–Lie–Trotter product formula.

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1. Introduction

The Zeno or 'a watched pot never boils' effect has, over the last few decades, attracted a lot of interest from quantum physicists, see [1–7] and the extensive list of references in [8]. The effect, which consists in the possible impedance of quantum evolution, e.g., decay processes, under the influence of frequent measurement events, or, more generally, frequent system–environment interactions, has even entered popular science texts [9]. It is a striking example of the peculiarities of quantum theory whose origin can be traced back to the geometry of the Hilbert space [10], which implies a quadratic short-time behaviour of transition probabilities [11].

Here, we rely on the mathematical formulation of the strict Zeno paradox as presented by Misra and Sudarshan in [12], i.e. the limit of infinitely frequent measurements, which we will now briefly sketch. Given the Hamiltonian evolution $U(t) = \mathrm{e}^{-\mathrm{i}Ht}$ on the Hilbert space $\mathcal{H}$, with H lower semibounded, and a projection E on $\mathcal{H}$, one assumes the existence of the limits

$$W(t) := \operatorname*{s\text{-}lim}_{n\to\infty} [EU(t/n)E]^n$$

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in the strong sense on $\mathcal{H}$, for all $t \in \mathbb{R}$ (this condition can be relaxed to $t \geqslant 0$ if a CPT-symmetry is present). It is then shown, using methods of complex analysis, that $W(t)$ is a strongly continuous group, $W(t+s) = W(t)W(s)$, for all $t \in \mathbb{R}$, and $W(-t) = W(t)^*$. In particular, one finds a lower semibounded operator B and a projection G such that $BG = GB = B$ which induces the Zeno dynamics: $W(t) = G\mathrm{e}^{-\mathrm{i}Bt}G$. If one employs the initial condition $\text{s-lim}_{t\to 0_+} W(t) = E$, one can identify G with E. Thus one sees that the Zeno dynamics is a modified Hamiltonian dynamics, which is confined to the Zeno subspace $E\mathcal{H}$.

Although the limit of infinitely frequent measurements has been argued to be unphysical due to Heisenberg uncertainty [8], it is still of conceptual interest. This is, in particular, the case if one wants to study the induced limiting dynamics $W(t)$ on the Zeno subspace and to compare it with the original one on the full space. This has, however, only been done for simple examples [1, 13]. It turns out that in these examples the Zeno dynamics corresponds to Hamiltonian evolution with additional constraints and boundary conditions. For example, one finds an infinite well potential term, corresponding to Dirichlet boundary conditions, when the projector is the multiplication with the characteristic function of an interval in the one-dimensional case.

In this paper we show how the treatment of [12] can be carried over to the modular flow of a von Neumann algebra $\mathcal{A}$. First and foremost, this is a direct generalization of the result of Misra and Sudarshan to dynamics whose generators are not lower semibounded. The special role projections play for von Neumann algebras gives our generalization some additional impact, in view of the ongoing discussion over the projection postulate. Furthermore, our analysis can also straightforwardly be applied to KMS states of W^*-dynamical systems for inverse temperatures $0 < \beta \leqslant \infty$. But what is, in our view, most important, is that our treatment yields an explicit identification of the Zeno dynamics: it can, in favourable cases, be shown to coincide with the unique modular flow of the localized von Neumann subalgebra $E\mathcal{A}E$. This result can be viewed as a variant of the Kato–Lie–Trotter product formula [14, corollary 3.1.31].

In the following section, we present this generalization of the strict Zeno paradox. The final section contains some remarks on the status of the main theorem 2.1, its weaknesses and possible extensions, as well as an outlook towards physical applications.

2. The Zeno paradox in the context of von Neumann algebras

Theorem 2.1. *Let $\mathcal{A}$ be a von Neumann algebra with faithful, normal state ω, represented on the Hilbert space $\mathcal{H}$ with cyclic and separating vector Ω associated with ω. Let Δ be the modular operator of $(\mathcal{A}, \Omega)$. Let $E \in \mathcal{A}$ be a projection. Set $\mathcal{A}_E := E\mathcal{A}E$ and define a subspace of $\mathcal{H}$ by $\mathcal{H}_E := \overline{\mathcal{A}_E\Omega} \subset E\mathcal{H}$. Assume:*

(i) for all $t \in \mathbb{R}$, the strong operator limits

$$W(t) := \operatorname*{s-lim}_{n\to\infty} [E\Delta^{\mathrm{i}t/n}E]^n$$

exist, are weakly continuous in t and satisfy the initial condition

$$\operatorname*{w-lim}_{t\to 0} W(t) = E.$$

(ii) for all $t \in \mathbb{R}$, the following limits exist:

$$W(t - \mathrm{i}/2) := \operatorname*{s-lim}_{n\to\infty} [E\Delta^{\mathrm{i}(t-\mathrm{i}/2)/n}E]^n$$

where the convergence is strong on the common, dense domain $\mathcal{A}\Omega$.

Then the $W(t)$ form a strongly continuous group of unitary operators on $\mathcal{H}_E$. The group $W(t)$ induces an automorphism group τ^E of $\mathcal{A}_E$ by

$$\tau^E : \mathcal{A}_E \ni A_E \longmapsto \tau_t^E(A_E) := W(t)A_E W(-t) = W(t)A_E W(t)^*$$

such that $(\mathcal{A}_E, \tau^E)$ is a W^*-dynamical system. The vectors $W(z)A_E\Omega$, $A_E \in \mathcal{A}_E$, are holomorphic in the strip $0 < -\mathrm{Im}\, z < 1/2$ and continuous on its boundary.

Note that $\mathcal{A}_E$ is a von Neumann subalgebra of $\mathcal{A}$, see [15, corollary 5.5.7], for which Ω is cyclic for $\mathcal{H}S_E$ and separating. Thus, Ω induces a faithful representation of $O\mathcal{A}_E$ on the closed Hilbert subspace $\mathcal{H}_E$, and thus all notions above are well defined.

Corollary 2.2. *Let Ω_E be a vector in $\mathcal{H}_E$ which induces a faithful, normal state ω_E on $\mathcal{A}_E$, and denote by Δ_E the modular operator of $(\mathcal{A}_E, \omega_E)$. Assume further the validity of the following additional condition:*
(iii) for all $A_E, B_E \in \mathcal{A}_E$ holds

$$\lim_{t\to 0} \langle W(-t - \mathrm{i}/2)A_E\Omega_E, W(t - \mathrm{i}/2)B_E\Omega_E\rangle = \left\langle \Delta_E^{1/2} A_E\Omega_E, \Delta_E^{1/2} B_E\Omega_E\right\rangle.$$

Then τ^E is the modular automorphism group of $(\mathcal{A}_E, \Omega_E)$.

The remainder of this section contains the proof of the above theorem and its corollary, split into several lemmata. In all these, we will only use conditions (i) and (ii). Only after that will condition (iii) be used to identify the modular group.

Set $S := \{z \in \mathbb{C} | -1/2 < \mathrm{Im}\, z < 0\}$. Define operator-valued functions

$$F_n(z) := [E\Delta^{\mathrm{i}z/n}E]^n \qquad \text{for} \quad z \in \overline{S} \qquad n \in \mathbb{N}.$$

The $F_n(z)$ are operators whose domains of definition contain the common, dense domain $\mathcal{A}\Omega$. They depend holomorphically on z in the sense that the vector-valued functions $F_n(z)A\Omega$ are holomorphic on S and continuous on $\overline{S}$ for every $A \in \mathcal{A}$. For this and the following lemma, see [14, sections 2.5 and 5.3 and theorem 5.4.4].

Lemma 2.3. *For $z \in \overline{S}$ and $\Psi \in D(\Delta^{|\mathrm{Im}\, z|})$ holds the estimate*

$$\|F_n(z)\Psi\| \leqslant \|\Psi\|$$

for all $n \in \mathbb{N}$.

Proof. Define vector-valued functions $f_k^{\Psi,n}(z) := [E\Delta^{\mathrm{i}z/n}E]^k\Psi$. These are well defined for $z \in \overline{S}, \Psi \in D(\Delta^{|\mathrm{Im}\, z|})$ and all $k \leqslant n$, since for such Ψ, z we have $[E\Delta^{\mathrm{i}z/n}E]^{k-1} \in D(E\Delta^{\mathrm{i}z/n}E)$. Approximate $f_{k-1}^{\Psi,n}(z)$ by elements of the form $A_l\Omega$, $A_l \in \mathcal{A}$. Then for any $B \in \mathcal{A}$ holds

$$\begin{aligned} |\langle B\Omega, E\Delta^{\mathrm{i}z/n}EA_l\Omega\rangle| &= |\langle \Omega, B^*E\Delta^{\mathrm{i}z/n}EA_l\Delta^{-\mathrm{i}z/n}\Omega\rangle| \\ &= |\omega(B^*E\sigma_{z/n}(EA_l))| \\ &\leqslant \|B^*E\Omega\|\|EA_l\Omega\| \\ &\leqslant \|B\|\|A_l\Omega\|. \end{aligned}$$

Here, ω is the state on $\mathcal{A}$ associated with the cyclic and separating vector Ω (we always identify elements of $\mathcal{A}$ with their representations on $\mathcal{H}$) and σ denotes the modular group. The first estimate above follows explicitly from the corresponding property of σ, see [14, proposition 5.3.7] (the connection between faithful states of von Neumann algebras and KMS states given by Takesaki's theorem [14, theorem 5.3.10] is used here and in the following).

This means $\|E\Delta^{\mathrm{i}z/n}EA_l\Omega\| \leqslant \|A_l\Omega\|$, and since $A_l\Omega \longrightarrow f_{k-1}^{\Psi,n}(z)$ in the norm of $\mathcal{H}$, it follows that $\|f_k^{\Psi,n}(z)\| \leqslant \|f_{k-1}^{\Psi,n}(z)\|$. Since this holds for all $k = 1, \ldots, n$, we see

$$\|F_n(z)\Psi\| = \|[t]f_n^{\Psi,n}(z)\| \leqslant \cdots \leqslant \|f_1^{\Psi,n}(z)\| \leqslant \|\Psi\|$$

as desired. □

The estimate proved above also yields that the F_n are closable. We will denote their closures by the same symbols in the following.

Lemma 2.4. *For $z \in S$ holds the representation*

$$F_n(z)A\Omega = \frac{(z+\mathrm{i})^2}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{F_n(t-\mathrm{i}/2)A\Omega}{(t+\mathrm{i}/2)^2(t-\mathrm{i}/2-z)} - \frac{F_n(t)A\Omega}{(t+\mathrm{i})^2(t-z)}\,\mathrm{d}t \qquad (1)$$

where the integrals are taken in the sense of Bochner. One further has

$$0 = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{F_n(t-\mathrm{i}/2)A\Omega}{(t+\mathrm{i}/2)^2(t-\mathrm{i}/2-z)} - \frac{F_n(t)A\Omega}{(t+\mathrm{i})^2(t-z)}\,\mathrm{d}t \qquad (2)$$

for $z \notin \overline{S}$.

Proof. By Cauchy's theorem for vector-valued functions [16, theorem 3.11.3], we can write

$$\frac{F_n(z)A\Omega}{(z+\mathrm{i})^2} = \frac{1}{2\pi\mathrm{i}} \oint \frac{F_n(\zeta)A\Omega}{(\zeta+\mathrm{i})^2(\zeta-z)}\,\mathrm{d}\zeta$$

where the integral runs over a closed, positively oriented contour in S, which encloses z. We choose this contour to be the boundary of the rectangle determined by the points $\{R-\mathrm{i}\varepsilon, -R-\mathrm{i}\varepsilon, -R-\mathrm{i}(1/2-\varepsilon), R-\mathrm{i}(1/2-\varepsilon)\}$ for $R > 0$, $1/4 > \varepsilon > 0$. By lemma 2.3, the norms of the integrals over the paths parallel to the real line stay bounded as $R \to \infty$, while those of the integrals parallel to the imaginary axis vanish. Thus

$$\frac{F_n(z)A\Omega}{(z+\mathrm{i})^2} = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{F_n(t-\mathrm{i}(1/2-\varepsilon))A\Omega}{(t+\mathrm{i}(1/2+\varepsilon))^2(t-\mathrm{i}(1/2-\varepsilon)-z)} - \frac{F_n(t)A\Omega}{(t+\mathrm{i}(1-\varepsilon))^2(t-\mathrm{i}\varepsilon-z)}\,\mathrm{d}t.$$

For $0 < \varepsilon_0 < \min\{|\mathrm{Im}\, z|, |1/2 - \mathrm{Im}\, z|\}$ and all ε such that $0 \leqslant \varepsilon \leqslant \varepsilon_0$, the integrand is bounded in norm by $\|A\|/[(1+t^2)\min\{|\mathrm{Im}\, z - \varepsilon_0|, |\mathrm{Im}\, z - (1/2-\varepsilon_0)|\}]$. Since moreover, in the strong sense and pointwise in t, $\lim_{\varepsilon\to 0} F_n(t-\mathrm{i}\varepsilon)A\Omega = F_n(t)A\Omega$ and $\lim_{\varepsilon\to 0} F_n(t - \mathrm{i}(1/2-\varepsilon))A\Omega = F_n(t-\mathrm{i}/2)A\Omega$, the conditions for the application of the vector-valued Lebesgue theorem on dominated convergence [16, theorem 3.7.9] are given and the desired representation follows in the limit $\varepsilon \to 0$. The vanishing of the second integral follows analogously. □

Lemma 2.5. *The strong limits $F(z) := \text{s-lim}_{n\to\infty} F_n(z)$, $z \in S$, are closable operators with common, dense domain $\mathcal{A}\Omega$ (we denote their closures by the same symbols). The integral representation*

$$F(z)A\Omega = \frac{(z+\mathrm{i})^2}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{W(t-\mathrm{i}/2)A\Omega}{(t+\mathrm{i}/2)^2(t-\mathrm{i}/2-z)} - \frac{W(t)A\Omega}{(t+\mathrm{i})^2(t-z)}\,\mathrm{d}t \qquad (3)$$

holds good and the functions $F(z)A\Omega$ are holomorphic on S, for all $A \in \mathcal{A}\Omega$. There exists a projection G and a positive operator Γ such that $\Gamma = G\Gamma = \Gamma G$ and $\Gamma^{4\mathrm{i}z} = F(z)$ for all $z \in S$.

Proof. Using lemma 2.3, we see that the norm of the integrand in equation (1) is uniformly bounded in n by $2\|A\|/[(1+t^2)\min\{|\mathrm{Im}\, z|, 1/2 - |\mathrm{Im}\, z|\}]$, which is integrable in t. Furthermore, $F_n(t)A\Omega$ and $F_n(t-\mathrm{i}/2)A\Omega$ converge in norm to $W(t)A\Omega$ and $W(t-\mathrm{i}/2)A\Omega$, respectively, by assumptions (i) and (ii) of theorem 2.1. Thus, we can again apply Lebesgue's theorem on dominated convergence to infer the existence of the limits $\lim_{n\to\infty} F_n(z)A\Omega$ for all $A \in \mathcal{A}$. This defines linear operators on the common, dense domain $\mathcal{A}\Omega$. Again, by the estimate of lemma 2.3, we have $F(z)A_n\Omega \to 0$ if $\|A_n\| \to 0$, and therefore the $F(z)$ are closable. The validity of equation (3) is then clear. Since the bound noted above is uniform in n and all the functions $F_n(z)A\Omega$ are holomorphic in S, we can apply the Stieltjes–Vitali theorem [16, theorem 3.14.1] to deduce the stated holomorphy of $F(z)A\Omega$. We now consider the operators $F(-\mathrm{i}s), 0 < s < 1/2$. Using the same properties of Δ, E, one sees that these operators are self-adjoint, and in fact, positive: namely, the limits are densely defined, symmetric and closable operators, and an analytic vector for $\Delta^{1/2}$ is also analytic for $F(-\mathrm{i}s), 0 < s < 1/2$. Thus the $F(-\mathrm{i}s)$ possess a common, dense set of analytic vectors. Under these circumstances, the $F(-\mathrm{i}s)$ are essentially self-adjoint, and we denote their unique, self-adjoint extension by the same symbol. We now follow [12] to show that the functional equation $F(-\mathrm{i}(s+t)) = F(-\mathrm{i}s)F(-\mathrm{i}t)$ holds for $s, t > 0$ such that $s+t < 1/2$. To this end, consider first the case that s and t are rationally related, i.e. there exist $p, q \in \mathbb{N}$ such that

$$\frac{s+t}{r(p+q)} = \frac{s}{rp} = \frac{t}{rq} \qquad \text{for all} \quad r \in \mathbb{N}.$$

Then

$$\left[E\Delta^{\frac{s+t}{r(p+q)}}E\right]^{r(p+q)} A\Omega = \left[E\Delta^{\frac{s}{rp}}E\right]^{rp}\left[E\Delta^{\frac{t}{rq}}E\right]^{rq} A\Omega \qquad A \in \mathcal{A}$$

from which the claim follows in the limit $r \to \infty$. The general case follows since $F(-\mathrm{i}s)A\Omega$ is holomorphic and, therefore, also strongly continuous in s for all $A \in \mathcal{A}$. Now set $\Gamma = F(-\mathrm{i}/4)$. By the spectral calculus for unbounded operators [15, section 5.6], the positive powers Γ^σ exist for $0 < \sigma \leqslant 1$, and are positive operators with domain containing the common, dense domain $\mathcal{A}\Omega$. They satisfy the functional equation $\Gamma^{\sigma+\tau} = \Gamma^\sigma \hat{\cdot}\, \Gamma^\tau$ for $\sigma, \tau > 0$ such that $\sigma + \tau \leqslant 1$, and where $\hat{\cdot}$ denotes the closure of the operator product. The solution to this functional equation with initial condition $\Gamma = F(-\mathrm{i}/4)$ is unique and thus follows $\Gamma^\sigma = F(-\mathrm{i}\sigma/4)$, since the operators F satisfy the same functional equation, and all operators in question depend continuously on σ, in the strong sense when applied to the common core $\mathcal{A}\Omega$. For $1/4 \leqslant s < 1/2$ we have $F(-\mathrm{i}s) = F(-\mathrm{i}/4)F(-\mathrm{i}(s-1/4)) = \Gamma F(-\mathrm{i}(s-1/4)) = \Gamma\Gamma^{4s-1} = \Gamma^{4s}$, which finally shows the identity $F(-\mathrm{i}s) = \Gamma^{4s}$ for $0 < s < 1/2$. Now, for every $A \in \mathcal{A}$, $\Gamma^{4\mathrm{i}z}A\Omega$ extends to a holomorphic function on S which coincides with $F(z)A\Omega$ on the segment $\{-\mathrm{i}s \mid 0 < s < 1/2\}$ as we have just seen. The identity theorem for vector-valued, holomorphic functions [16, theorem 3.11.5] then implies $\Gamma^{4\mathrm{i}z}A\Omega = F(z)A\Omega, z \in S$ and all $A \in \mathcal{A}$. Thus $\Gamma^{4\mathrm{i}z} = F(z)$ holds on S as an identity of densely defined, closed operators. Setting $G = P([0,\infty))$, where P is the spectral resolution of the identity for Γ, we see that we can write $\Gamma = G\Gamma = \Gamma G$, concluding this proof. □

Lemma 2.6. *It holds $G = E$ and $W(t) = E\Gamma^{4\mathrm{i}t}E$, for all t.*

Proof. Using (3) we can write, adding a zero contribution to that integral representation,

$$\langle B\Omega, F(t-\mathrm{i}\eta)A\Omega\rangle = \frac{(t+\mathrm{i}-\mathrm{i}\eta)^2}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}s\left\{\frac{\langle B\Omega, W(s-\mathrm{i}/2)A\Omega\rangle}{(s+\mathrm{i}/2)^2(s-t-\mathrm{i}/2+\mathrm{i}\eta)}\right.$$
$$\left. - \frac{\langle B\Omega, W(s)A\Omega\rangle}{(s+\mathrm{i})^2(s-t+\mathrm{i}\eta)} - \frac{\langle B\Omega, W(s-\mathrm{i}/2)A\Omega\rangle}{(s+\mathrm{i}/2)^2(s-t-\mathrm{i}/2-\mathrm{i}\eta)} + \frac{\langle B\Omega, W(s)A\Omega\rangle}{(s+\mathrm{i})^2(s-t-\mathrm{i}\eta)}\right\}$$

where the integral over the last two terms is zero, as can be seen from (2) and the same arguments that were used to derive (3). This yields

$$\langle B\Omega, F(t-\mathrm{i}\eta)A\Omega\rangle = \frac{(t+\mathrm{i}-\mathrm{i}\eta)^2}{\pi}\int_{-\infty}^{\infty} \frac{\eta\cdot\langle B\Omega, W(s-\mathrm{i}/2)A\Omega\rangle}{(s+\mathrm{i}/2)^2((s-t-\mathrm{i}/2)^2+\eta^2)} - \frac{\eta\cdot\langle B\Omega, W(s)A\Omega\rangle}{(s+\mathrm{i})^2((s-t)^2+\eta^2)}\,\mathrm{d}s.$$

As $\eta \to 0_+$, the first term under the integral vanishes, while the second reproduces the integrable function $\langle A\Omega, W(t)B\Omega\rangle/(t+\mathrm{i})^2$ as the boundary value of its Poisson transformation. Thus we have seen

$$\lim_{\eta\to 0_+} \langle A\Omega, F(t-\mathrm{i}\eta)B\Omega\rangle = \langle A\Omega, W(t)B\Omega\rangle$$

for given $A, B \in \mathcal{A}$, and almost all $t \in \mathbb{R}$. Since the integral is uniformly bounded in η, the boundary value of this Poisson transformation is continuous in t, see, e.g., [17, section 5.4]. The same holds for $\langle A\Omega, W(t)B\Omega\rangle$ by assumption (i) of theorem 2.1, and therefore the limiting identity at $\eta = 0$ follows for all t. On the other hand, since $G\Gamma^{4\mathrm{i}t}G$ is strongly continuous in t, we have $\lim_{\eta\to 0_+} \langle A\Omega, F(t-\mathrm{i}\eta)B\Omega\rangle = \langle A\Omega, G\Gamma^{4\mathrm{i}t}GB\Omega\rangle$ for all t. Thus, the identity of bounded operators $W(t) = G\Gamma^{4\mathrm{i}t}G$ holds for all t. By assumption we have $\text{w-lim}_{t\to 0} W(t) = E$, thus $W(s)W^*(s) = G\Gamma^{4\mathrm{i}s}\Gamma^{-4\mathrm{i}s}G = G$ implies $G = E$. □

Lemma 2.7. *The action* $\tau_t^E : \mathcal{A}_E \ni A_E \longmapsto \tau_t^E(A_E) = \Gamma^{4\mathrm{i}t}A_E\Gamma^{-4\mathrm{i}t}$ *is a strongly continuous group of automorphisms of* $\mathcal{A}_E$.

Proof. For $A_E = EAE \in \mathcal{A}_E$ we have $E\Delta^{\mathrm{i}t/n}EA_EE\Delta^{-\mathrm{i}t/n}E = E\sigma_{t/n}(A_E)E$, where σ is the modular group of $(\mathcal{A}, \Omega)$, and this shows $F_n(t)A_EF_n(-t) \in \mathcal{A}_E$ for all n. Since $\mathcal{A}_E$ is weakly closed and $F_n(t)A_EF_n(-t)$ converges strongly, and therefore also weakly, by assumption (i) of theorem 2.1, it converges to an element of $\mathcal{A}_E$. Since $\|F_n(t)A_EF_n(-t)\| \leqslant \|A_E\|$ for all n, the limit mapping is continuous on $\mathcal{A}_E$. By lemma 2.6, it equals τ_t^E, as defined above, for all t. Since $\Gamma^{4\mathrm{i}t}$ is a strongly continuous group of unitary operators on $E\mathcal{H}$, the assertion follows. □

Proof of theorem 2.1 and corollary 2.2. We first note that $W(-t) = W(t)^*$ can be seen by direct methods as in [12]. Secondly, since τ^E is an automorphism group of $\mathcal{A}_E$, it follows by definition of $\mathcal{H}_E$ that the $W(t)$ leave that subspace invariant and thus form a unitary group on it. The stated analyticity properties of W are contained in the conclusions of lemmata 2.5 and 2.6.

Let us now turn to the identification of W with the modular group of the pair $(\mathcal{A}_E, \Omega_E)$. An argument as was used in the proof of lemma 2.6 shows

$$\langle A\Omega, \Gamma^{4\mathrm{i}(t-\mathrm{i}/2)}B\Omega\rangle = \lim_{\eta\to 1/2_-} \langle A\Omega, F(t-\mathrm{i}\eta)B\Omega\rangle = \langle A\Omega, W(t-\mathrm{i}/2)B\Omega\rangle$$

for given $A, B \in \mathcal{A}$, and almost all $t \in \mathbb{R}$. Additionally, as mentioned in the proof of lemma 2.6, the boundary value of the Poisson transformation $\Gamma^{4\mathrm{i}(t-\mathrm{i}/2)}$ is weakly continuous on $\mathcal{A}\Omega$. From this, the density of $\mathcal{A}\Omega$ in $\mathcal{H}$, and assumption (iii) of theorem 2.1, it follows for $A_E, B_E \in \mathcal{A}_E$ that

$$\begin{aligned}\langle \Gamma^2 A_E\Omega_E, \Gamma^2 A_E\Omega_E\rangle &= \lim_{t\to 0} \langle \Gamma^{4\mathrm{i}(-t-\mathrm{i}/2)}B_E\Omega_E, \Gamma^{4\mathrm{i}(t-\mathrm{i}/2)}A_E\Omega_E\rangle \\ &= \lim_{t\to 0} \langle W(-t-\mathrm{i}/2)A_E\Omega_E, W(t-\mathrm{i}/2)A_E\Omega_E\rangle \\ &= \langle \Delta^{1/2}A_E\Omega_E, \Delta^{1/2}A_E\Omega_E\rangle\end{aligned}$$

where the weak continuity has been used in the first and the identity above in the second step. Now by the modular condition satisfied by Δ, this becomes

$$\langle \Gamma^2 A_E \Omega_E, \Gamma^2 A_E \Omega_E \rangle = \langle B_E^* \Omega_E, A_E^* \Omega_E \rangle.$$

This is the modular condition for the automorphism group τ^E with respect to $(\mathcal{A}_E, \Omega_E)$. The assertion of the theorem follows by the uniqueness of the modular group [15, theorem 9.2.16] and the preceding three lemmata. □

3. Conclusions

Let us comment a bit on the status of theorem 2.1. It is stronger than the result of [12] in that it generalizes it to the KMS states of W^*-dynamical systems at inverse temperatures $0 < \beta \leqslant \infty$. This is exactly the framework in which a strip of analyticity of width β above (or below, depending on convention) the real axis exists, which is the sole condition needed to apply the methods of complex analysis used extensively to prove theorem 2.1. This shows that the theorem of Misra and Sudarshan [12] can be extended to the cases in which the Hamiltonian is not lower semibounded, but in which its negative spectral parts are 'exponentially damped' (see [18, section V.2.1] for the precise meaning of these notions). This is in contrast to the counterexample in [12], where those authors state that lower semiboundedness is essential. That counterexample involves the unitary group generated by the momentum operator, which does not fulfil any requirement of exponential damping of the negative spectral part, and thus violates our analyticity assumptions. See also the discussion in [3, section 3], where it is noted that lower semiboundedness does not seem to be important for the Zeno effect in general.

But more interestingly, corollary 2.2 identifies the induced Zeno dynamics uniquely, as already mentioned in the introduction. For this, however, we needed the additional assumption (iii) in corollary 2.2, and this assumption is not a simple consequence of the modular condition of the original system. But note that we did not use assumption (iii) until after lemma 2.7, so that in any case we get an automorphic Zeno dynamics on $\mathcal{A}_E$, which however might differ from the modular dynamics, depending on the choice of state on $\mathcal{A}_E$, but has the analyticity properties needed to check the modular condition for a given state as in corollary 2.2. Assumption (iii) is therefore only a formal condition for Ω_E to be a τ^E-KMS state, essentially a straightforward transcription of the modular condition itself. One simple example of an equilibrium state for the Zeno dynamics can be given in the case when the projector E commutes with the initial dynamics, i.e. $[E, \Delta] = 0$. Then the Zeno dynamics simplifes to $W(t) = E\Delta^{\mathrm{i}t}E$ and the state $\Omega_E := E\Omega/\|E\Omega\|$ satisfies condition (iii). Other, less trivial, examples can arise from Gibbs equilibria, as is shown in [19].

As already remarked in [12], the relatively strong assumptions of the theorem might be difficult to prove in concrete cases. It seems more likely that in studying physical models one would rather identify the nature of the induced Zeno dynamics directly, as for example in [1, 13]. Therefore, theorem 2.1 and corollary 2.2 are to be considered as a mathematical gedanken experiment, which might be helpful in guiding physical intuition.

However, some relaxations of the assumptions of theorem 2.1 are possible in special cases [12]: firstly, if the theory contains a CPT-operator, the conclusions of theorem 2.1 already follow if one assumes the convergence of the limits defining $W(t)$, $W(t - \mathrm{i}/2)$ only for positive times. Secondly, if we restrict our attention to von Neumann algebras $\mathcal{A}$ that allow a faithful representation on a *separable* Hilbert space $\mathcal{H}$, we can drop the assumption of weak continuity of the limits $W(t)$, $W(t - \mathrm{i}/2)$ used in the proof of lemma 2.6, by the argument given in [12]: first show that $\lim_{\eta \to 0_+} \langle A\Omega, F(t - \mathrm{i}\eta)B\Omega \rangle = \langle A\Omega, W(t)B\Omega \rangle$ for t outside an exceptional null set $\mathcal{N}_{A,B} \subset \mathbb{R}$. Then, there exists a countable set $\mathcal{C} \subset \mathcal{A}$ such that $\mathcal{C}\Omega$ is dense in $\mathcal{H}$ and the countable union $\bigcup_{A,B\in\mathcal{C}} \mathcal{N}_{A,B}$ is still a null set. One easily shows that the limit relation holds true outside this set in the weak sense on $\mathcal{H}$ and proceeds from there using the strong continuity of $G\Gamma^{4it}G$ as in [12].

An application to simple examples of quantum statistical mechanics, such as spin systems, seems possible. There, if the projection E projects to some pure state outside a bounded region, one is in a case where one would presume the Zeno dynamics to exist. One could expect to find the dynamics of the bounded region of the spin system (a matrix model) with appropriate boundary conditions, to be determined by the action of E on the boundary layer (see [20] for a complementary discussion). This and further applications in the context of quantum statistical mechanics are found in a subsequent paper [19].

There will also appear a subtle point in physical applications: when considering models one usually deals with C^*-dynamical systems rather than W^*-dynamical ones, i.e. the relevant algebras are norm rather than weakly closed. For these, the set of KMS states at given inverse temperature is in general a nonempty, weak $*$-closed, convex subset of the state space [14, section 5.3.2]. Thus, one always has to choose a KMS state and an associated representation to work within. If one fixes a KMS state, say ω with vector representative Ω, and wants to consider the induced Zeno dynamics determined by a projector E, the restricted subsytem $\mathcal{A}_E$ will still have in general a multitude of KMS states of its own. The point is that it is not *a priori* clear that the Zeno dynamics will leave a chosen state ω_E invariant. It may happen that the Zeno dynamics transforms the KMS states of $\mathcal{A}_E$ into each other in a nontrivial way. In this case the Zeno dynamics is not unitary, i.e. reversible. This is a reflection of the same problem appearing in the quantum mechanical context [1, section 5]. This cannot happen, however, within the context of theorem 2.1, for if the Zeno dynamics satisfies condition (iii), then invariance of Ω_E follows directly [14, proposition 5.3.3]. But even if the equilibrium condition (iii) of that theorem is not satisfied, lemma 2.7 still assures that the induced dynamics is unitary. For these problems, see also the recent approach of Gustafson [7], who tackles the associated problem of self-adjointness of the generator of the Zeno dynamics (in [7], one also finds some important remarks on the history of the subject).

Finally, we remark that we were in part motivated by the proposal to take the modular flow of observable algebras as a definition of physical time [21–23], for example on a generally relativistic background, when the usual concepts are less useful. However, the relation between modular groups and spacetime is an intricate one, as local quantum physics has taught us [24, 25], and this thread of thought might be just beginning.

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